

PROCEEDINGS OF THE SYMPOSIUM ON
THE FOUNDATIONS OF MATHEMATICS

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Errata
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Page 13, line \uparrow 1:

$$P \in \Delta_R, \text{car}(P) \simeq p \rightarrow m : n, Q \in \Delta_N, \text{car}(Q) \simeq m + n \rightarrow k \Rightarrow [p' \rightarrow k : n; P; Q] \in \Delta_R$$

Page 14, line 2:

$$P \in \Delta_R, \text{car}(P) \simeq m : n \rightarrow p, Q \in \Delta_N, \text{car}(Q) \simeq m + n \rightarrow k \Rightarrow [k : n \rightarrow p'; P; Q] \in \Delta_R$$

Syntactic Construction of Rational Numbers

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Abstract Syntax

$\langle \text{natnum} \rangle = \langle \text{one} \rangle + \langle \text{natnum} \rangle \langle \text{prime} \rangle : i, j, k, l, m, n$
 $\langle \text{positive} \rangle = \langle \text{small} \rangle + \langle \text{medium} \rangle + \langle \text{large} \rangle : p, q, r$
 $\langle \text{small} \rangle = \langle \text{large} \rangle \langle \text{star} \rangle$
 $\langle \text{medium} \rangle = \langle \text{one} \rangle$
 $\langle \text{large} \rangle = \langle \text{positive} \rangle \langle \text{prime} \rangle$
 $\langle \text{rational} \rangle = \langle \text{negative} \rangle + \langle \text{zero} \rangle + \langle \text{positive} \rangle : u, v, w$
 $\langle \text{negative} \rangle = \langle \text{minus} \rangle \langle \text{positive} \rangle$
 $\langle \text{ratio} \rangle = \langle \text{natnum} \rangle \langle \text{colon} \rangle \langle \text{natnum} \rangle$
 $\langle \text{Nform} \rangle = \langle \text{one} \rangle + \langle \text{Nform} \rangle \langle \text{prime} \rangle + \langle \text{Nform} \rangle \langle \text{plus} \rangle \langle \text{Nform} \rangle + \langle \text{Nform} \rangle \langle \text{minus} \rangle \langle \text{Nform} \rangle$
 $\quad + \langle \text{Nform} \rangle \langle \text{times} \rangle \langle \text{Nform} \rangle : M, N$
 $\langle \text{Pform} \rangle = \langle \text{one} \rangle + \langle \text{Pform} \rangle \langle \text{prime} \rangle + \langle \text{Pform} \rangle \langle \text{star} \rangle + \langle \text{Pform} \rangle \langle \text{plus} \rangle \langle \text{Pform} \rangle$
 $\quad + \langle \text{Pform} \rangle \langle \text{minus} \rangle \langle \text{Pform} \rangle + \langle \text{Pform} \rangle \langle \text{times} \rangle \langle \text{Pform} \rangle$
 $\quad + \langle \text{Pform} \rangle \langle \text{slash} \rangle \langle \text{Pform} \rangle : P, Q$
 $\langle \text{Qform} \rangle = \langle \text{one} \rangle + \langle \text{zero} \rangle + \langle \text{Qform} \rangle \langle \text{prime} \rangle + \langle \text{Qform} \rangle \langle \text{star} \rangle + \langle \text{Qform} \rangle \langle \text{plus} \rangle \langle \text{Qform} \rangle$
 $\quad + \langle \text{Qform} \rangle \langle \text{minus} \rangle \langle \text{Qform} \rangle + \langle \text{Qform} \rangle \langle \text{times} \rangle \langle \text{Qform} \rangle + \langle \text{Qform} \rangle \langle \text{slash} \rangle \langle \text{Qform} \rangle$
 $\quad : U, V$
 $\langle \text{one} \rangle := 1$
 $\langle \text{prime} \rangle := '$
 $\langle \text{star} \rangle := *$
 $\langle \text{minus} \rangle := -$
 $\langle \text{colon} \rangle := :$
 $\langle \text{times} \rangle := \cdot$
 $\langle \text{slash} \rangle := /$
 $\langle \text{zero} \rangle := 0$
 $\langle \text{plus} \rangle := +$

Formal Systems

$\Delta : P, Q, R, S, T$

Δ_N

$[n \rightarrow n], [n+1 \rightarrow n'], [n' - n \rightarrow 1], [n \cdot 1 \rightarrow n] \in \Delta_N$
 $P \in \Delta_N, \text{car}(P) \approx N \rightarrow n \Rightarrow [N' \rightarrow n'; P] \in \Delta_N$
 $P \in \Delta_N, \text{car}(P) \approx n+m \rightarrow k \Rightarrow [n+m' \rightarrow k'; P] \in \Delta_N$
 $P \in \Delta_N, \text{car}(P) \approx n-m \rightarrow k \Rightarrow [n'-m \rightarrow k'; P] \in \Delta_N$
 $P, Q \in \Delta_N, \text{car}(P) \approx n \cdot m \rightarrow k, \text{car}(Q) \approx k+m \rightarrow l \Rightarrow [n \cdot m' \rightarrow l; P; Q] \in \Delta_N$
 $P, Q, R \in \Delta_N, \text{car}(P) \approx N \rightarrow n, \text{car}(Q) \approx M \rightarrow m, \text{car}(R) \approx n @ m \rightarrow k$
 $\Rightarrow [N @ M \rightarrow k; P; Q; R] \in \Delta_N, \text{ where } @ \approx + \text{ or } - \text{ or } \cdot$

Δ_R

$P \in \Delta_N \Rightarrow P \in \Delta_R$
 $[1 \rightarrow 1:1], [n: n \rightarrow 1] \in \Delta_R$
 $P \in \Delta_R, \text{car}(P) \approx p \rightarrow m: n, Q \in \Delta_N, \text{car}(Q) \approx m+n \rightarrow k \Rightarrow [p' \rightarrow k: n; P] \in \Delta_R$

$$\begin{aligned}
P \in \Delta_R, \text{car}(P) \approx p' \rightarrow m:n &\Rightarrow [p' \rightarrow m:n; P] \in \Delta_R \\
P \in \Delta_R, \text{car}(P) \approx m:n \rightarrow k, Q \in \Delta_N, \text{car}(Q) \approx m+n \rightarrow k &\Rightarrow [k:n \rightarrow p'; P] \in \Delta_R \\
P \in \Delta_R, \text{car}(P) \approx m:n \rightarrow p' &\Rightarrow [n:m \rightarrow p'; P] \in \Delta_R
\end{aligned}$$

Δ_P

$$\begin{aligned}
\{p \rightarrow p\} &\in \Delta_P \\
P \in \Delta_P, \text{car}(P) \approx P \rightarrow p &\Rightarrow [P' \rightarrow p'; P] \in \Delta_P \\
P \in \Delta_P, \text{car}(P) \approx P \rightarrow p^* &\Rightarrow [P^* \rightarrow p; P] \in \Delta_P \\
P \in \Delta_P, \text{car}(P) \approx P \rightarrow 1 &\Rightarrow [P^* \rightarrow 1; P] \in \Delta_P \\
P \in \Delta_P, \text{car}(P) \approx P \rightarrow p' &\Rightarrow [P^* \rightarrow p'^*; P] \in \Delta_P \\
P, Q, T \in \Delta_R, R, S \in \Delta_N, \text{car}(P) \approx p \rightarrow k:l, \text{car}(Q) \approx q \rightarrow m:n, \text{car}(R) \approx k \cdot n \pm l \cdot m \rightarrow i, \\
\text{car}(S) \approx l \cdot n \rightarrow j, \text{car}(T) \approx i:j \rightarrow r &\Rightarrow [p \pm q \rightarrow r; P; Q; R; S; T] \in \Delta_P \\
P, Q, T \in \Delta_R, R, S \in \Delta_N, \text{car}(P) \approx p \rightarrow k:l, \text{car}(Q) \approx q \rightarrow m:n, \text{car}(R) \approx k \cdot m \rightarrow i, \\
\text{car}(S) \approx l \cdot n \rightarrow j, \text{car}(T) \approx i:j \rightarrow r &\Rightarrow [p \cdot q \rightarrow r; P; Q; R; S; T] \in \Delta_P \\
P, Q, T \in \Delta_R, R, S \in \Delta_N, \text{car}(P) \approx p \rightarrow k:l, \text{car}(Q) \approx q \rightarrow m:n, \text{car}(R) \approx k \cdot n \rightarrow i, \\
\text{car}(S) \approx l \cdot m \rightarrow j, \text{car}(T) \approx i:j \rightarrow r &\Rightarrow [p/q \rightarrow r; P; Q; R; S; T] \in \Delta_P \\
P, Q, R \in \Delta_P, \text{car}(P) \approx P \rightarrow p, \text{car}(Q) \approx Q \rightarrow q, \text{car}(R) \approx p @ q \rightarrow r \\
\Rightarrow [P @ Q \rightarrow r; P; Q; R] &\in \Delta_P, \text{ where } @ \approx + \text{ or } - \text{ or } \cdot \text{ or } /.
\end{aligned}$$

Δ_Q (Axioms and rules for $'$, $*$ and $-$ are omitted here.)

$$\begin{aligned}
P \in \Delta_P &\Rightarrow P \in \Delta_Q \\
\{u \rightarrow u\}, \{u+0 \rightarrow u\}, \{0+u \rightarrow u\}, \{p+(-p) \rightarrow 0\}, \{(-p)+p \rightarrow 0\}, \{u \cdot 0 \rightarrow 0\}, \{0 \cdot u \rightarrow 0\} &\in \Delta_Q \\
P \in \Delta_P, \text{car}(P) \approx p-q \rightarrow r &\Rightarrow [p+(-q) \rightarrow r; P], [(-p)+q \rightarrow r; P] \in \Delta_Q \\
P \in \Delta_P, \text{car}(P) \approx q-p \rightarrow r &\Rightarrow [p+(-q) \rightarrow r; P], [(-p)+q \rightarrow r; P] \in \Delta_Q \\
P \in \Delta_P, \text{car}(P) \approx p \cdot q \rightarrow r &\Rightarrow [p \cdot (-q) \rightarrow r; P], [(-p) \cdot q \rightarrow r; P], [(-p) \cdot (-q) \rightarrow r; P] \in \Delta_Q \\
P, Q, R \in \Delta_Q, \text{car}(P) \approx U \rightarrow u, \text{car}(Q) \approx V \rightarrow v, \text{car}(R) \approx u @ v \rightarrow w \\
\Rightarrow [U @ V \rightarrow w; P; Q; R] &\in \Delta_Q, \text{ where } @ \approx + \text{ or } \cdot \\
P \in \Delta_Q, \text{car}(P) \approx U+(-V) \rightarrow w &\Rightarrow [U-V \rightarrow w; P] \in \Delta_Q \\
P \in \Delta_Q, \text{car}(P) \approx U \cdot V \rightarrow w &\Rightarrow [U/V \rightarrow w; P] \in \Delta_Q \\
\text{Notations: } P \vdash U \rightarrow u &\Leftrightarrow P \in \Delta_Q, \text{car}(P) \approx U \rightarrow u \\
\vdash U \rightarrow u &\Leftrightarrow \exists P \in \Delta_Q, P \vdash U \rightarrow u \\
U = V &\Leftrightarrow \exists u (\vdash U \rightarrow u \text{ and } \vdash V \rightarrow u)
\end{aligned}$$

Theorem The set <rational> enjoys the usual properties of the rational number field Q .