Toward a modalized linear-non-linear model

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Modal linear logic, a linear-logical reconstruction of intuitionistic modal logic S4 [F. & Yoshimizu 2019]

Modal linear logic is an integration of modal logic and linear logic, with a modality $\boxed{!}$, an integration of the $\Box$-modality and the !-exponential.

Formula

$A, B ::= p \mid A \multimap B \mid\ !A \mid \boxed{!}A$

Girard translation from Int. S4

$$\lceil p \rceil \stackrel{\text{def}}{=} p$$
$$\lceil A \to B \rceil \stackrel{\text{def}}{=} (!\lceil A \rceil) \multimap \lceil B \rceil$$
$$\lceil \Box A \rceil \stackrel{\text{def}}{=} \boxed{!}\lceil A \rceil$$

Inference rule

$$\frac{}{A \vdash A}\text{Ax} \quad \frac{\Gamma \vdash A \quad A, \Gamma' \vdash B}{\Gamma, \Gamma' \vdash B}\text{Cut} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}{\multimap}\text{R} \quad \frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \multimap B \vdash C}{\multimap}\text{L}$$

$$\frac{\boxed{!}\Delta, !\Gamma \vdash A}{\boxed{!}\Delta, !\Gamma \vdash !A}!\text{R} \quad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}!\text{L} \quad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}!\text{W} \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}!\text{C}$$

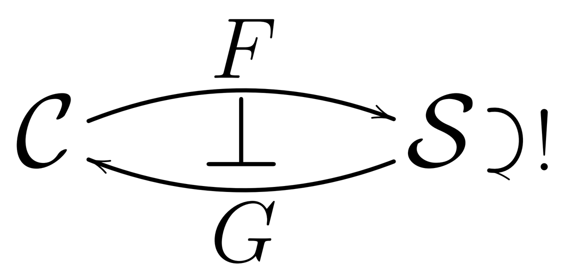
$$\frac{\boxed{!}\Delta \vdash A}{\boxed{!}\Delta \vdash \boxed{!}A}\boxed{!}\text{R} \quad \frac{\Gamma, A \vdash B}{\Gamma, \boxed{!}A \vdash B}\boxed{!}\text{L} \quad \frac{\Gamma \vdash B}{\Gamma, \boxed{!}A \vdash B}\boxed{!}\text{W} \quad \frac{\Gamma, \boxed{!}A, \boxed{!}A \vdash B}{\Gamma, \boxed{!}A \vdash B}\boxed{!}\text{C}$$

Thm 1. Soudness of the Girard translation; 2. Cut-elimination theorem; 3. Conservative extension to linear logic.

The aim of this work

Aim To create a categorical semantics of modal linear logic by means of *linear-non-linear adjunction*

Linear-non-linear model [Benton 1996]



- $\mathcal{C}$: Cartesian closed category
- $\mathcal{S}$: Symmetric monoidal closed category
- $F \dashv G$: Symmetric monoidal adjunction

These data yield a linear exponential comonad !, defined as $! \stackrel{\text{def}}{=} FG$, that characterizes the structure of !-exponential

Thm Intuitionistic multiplicative exponential linear logic is interpreted in $\mathcal{S}$ using !

Modal category theory [Kavvos 2017] (and others)

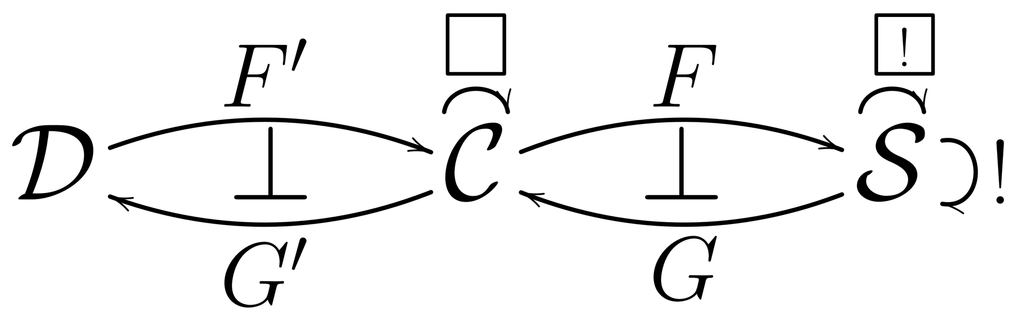
$$\mathcal{C} \circlearrowright \Box$$

- $\mathcal{C}$: Cartesian closed category
- $\Box$: Product-preserving functor (i.e., $\Box(A \wedge B) \cong \Box A \wedge \Box B$) with an additional condition depending on the logic, e.g.,
  - $\Box$: No additional condition (if the logic is K)
  - $\Box$: "Half a comonad" (if the logic is K4)
  - $\Box$: Equipped w/ a nat. trans. $\varepsilon : \Box \Rightarrow \mathrm{Id}$ (if the logic is T)
  - $\Box$: Comonad (if the logic is S4)

Thm Intuitionistic ($\Box$-fragment of) modal logics are interpreted in the above structure

Modal LNL model (for S4 modal linear logic)

Idea The idea to create a model of modal linear logic is to combine the model of linear logic and that of modal logic



- $\mathcal{D}, \mathcal{C}$: Cartesian closed category
- $\mathcal{S}$: Symmetric monoidal closed category
- $F' \dashv G', F \dashv G$: Symmetric monoidal adjunction†

These data yield two linear exponential comonads $\boxed{!}$ and !, which are defined as $FF'G'G$ and FG, respectively

† The idea to use the adjunction $F' \dashv G'$ of the modal part is suggested by Shin-ya Katsumata to the author

Property of the modal LNL model

- **Thm** The modal part ($\mathcal{D} \rightleftarrows \mathcal{C}$) models intuitionistic modal logic S4
- **Thm** The linear part ($\mathcal{C} \rightleftarrows \mathcal{S}$) models intuitionistic multiplicative exponential linear logic
- **Thm** The whole modal LNL model ($\mathcal{D} \rightleftarrows \mathcal{C} \rightleftarrows \mathcal{S}$) models intuitionistic modal linear logic

On-going work & future direction

- To define "modal linear category" which is a something analogous to the so-called *linear category*
- To use the model to analyze the computational structure of $\lambda^{\boxed{!}}$ [F. & Yoshimizu 2019], the typed λ-calc. of modal linear logic
- To generalize the model to cover other modal logics and linear logics, other than the pair of (S4, MELL)